A Complete Solution of the Inductive Iris with TE_{k0} Incidence in Rectangular Waveguide

JOSEPH C. PALAIS, MEMBER, IEEE

Abstract—A TE_{k0} wave incident on an inductive metal iris in a rectangular waveguide excites a reflected and a transmitted wave in the TE_{k0} mode. The reflection and transmission coefficients describing these waves are computed. In addition to the incident mode, a number of other modes are excited by the discontinuity. The amount of coupling to these other modes, given by coupling coefficients, is determined using the variational technique. The method developed makes it possible to find the coupling to any desired mode without first finding the coupling to any other mode or group of modes. The analysis shows under what conditions certain modes can be suppressed or eliminated. The method should be applicable to other problems of interest where modes other than the incident one are excited.

Since the reflection, transmission, and coupling coefficients are known, the total field at any point in the waveguide can be computed. As an example, the total field at the discontinuity when the TE_{10} mode is incident is calculated. The result closely resembles the expected result (of zero electric field over the metal iris).

I. Introduction

The inductive metal iris with zero thickness in rectangular waveguide (see Fig. 1) has been extensively studied and documented [1]–[5]. In the reports just cited, finding the reflection coefficient when the TE_{10} mode was incident was the main goal. It was found in these derivations that the problem did not have to be solved completely (that is, the amount of excitation of higher-order modes did not have to be found) to obtain the reflection coefficient. Basically, this paper derives relatively simple expressions for the coupling coefficients. These coefficients give the amount of coupling to any of the higher-order modes. The entire field in the aperture (or indeed anywhere in the waveguide) can be found from this information. In addition, the analysis considers the case where higher-order TE_{k0} modes are incident and derives their coupling to other modes (lower-order as well as higher-order). This information is useful when two or more waveguide discontinuities are close together and the various modes excited by one obstacle act upon the other obstacle. Collin, while considering closely spaced capacitive irises, calculated the reflection and coupling coefficients for two interacting modes [4].

The results of this paper are applicable to the case of propagation in oversize rectangular waveguide where modes other than the TE_{10} mode can propagate. Oversize waveguide is often used at millimeter wavelengths and for high-power devices at longer wavelengths.

II. Reflection Coefficients

The infinitely thin iris in a rectangular waveguide is shown in Fig. 1. The TE_{k0} mode is incident so that only the y-component of the electric field exists in the waveguide and the fields are independent of the y-coordinate. Thus, only the set of TE_{l0} modes are included in the solution. A time dependence $e^{j\omega t}$ is assumed. The electric field is given by

$$E_k = \sin\frac{k\pi x}{a}e^{-\gamma_k z} + \sum_{l=1}^{\infty} R_{kl}\sin\frac{l\pi x}{a}e^{\gamma_l z} \quad z < 0 \tag{1}$$

$$E_k = \sum_{l=1}^{\infty} T_{kl}\sin\frac{l\pi x}{a}e^{-\gamma_l z} \quad z > 0 \tag{2}$$

where

$$\gamma_l^2 = \left(\frac{l\pi}{a}\right)^2 - k_0^2 \quad \text{and} \quad k_0^2 = \omega^2\mu\epsilon.$$

The quantity R_{kk} is the reflection coefficient. It represents the relative amplitude of the wave reflected in the same mode as the incident wave. The R_{kl} terms (with $l\neq k$) are called the coupling coefficients and represent excitation of modes different from the incident one. The T_{kl} terms are the usual transmission coefficients. R_{11} has been computed in references [1]–[5]. R_{kk}, for $k\geq 1$ will be computed here by a method similar to that used by Collin [6] to find R_{11}. The R_{kl} terms will be calculated in Section III.

Since the electric field is continuous at $z=0$, we have $1+R_{kk}=T_{kk}$ and $R_{kl}=T_{kl}$ for $l\neq k$. Thus, a determination of the R_{kk} and R_{kl} terms solves the problem completely. The infinitely thin iris can be represented by a pure shunt admittance (see Fig. 2). This admittance across a transmission line with unit characteristic impedance is related to the reflection coefficient produced by $Y_k = -2R_{kk}/(1+R_{kk})$.

A variational expression for the admittance of the iris when the TE_{k0} mode is incident is found to be

$$Y_k = \frac{\dfrac{2}{\gamma_k}\sum_{l=1}^{\infty}{}' \gamma_l\left[\int_{AP} E_k(x)\sin\frac{l\pi x}{a}\,dx\right]^2}{\left[\int_{AP} E_k(x)\sin\frac{k\pi x}{a}\,dx\right]^2} \tag{3}$$

where the prime denotes the $l=k$ term is to be omitted in the summation. This equation is similar to that obtained by Collin for the case of TE_{10} incidence [7]. The integrations extend over the aperture.

Integration by parts changes the trial function from $E_k(x)$ to its derivative in (3). The summation in the resulting expression can be broken into a part that can be explicitly summed and a part that converges quickly. After some manipulation the resultant variational expression for the admittance is

Manuscript received February 2, 1966; revised October 21, 1966.
The author is at the College of Engineering Sciences, Arizona State University, Tempe, Ariz.

$$
Y_k = \frac{\begin{aligned}&\frac{2}{\gamma_k}\left\{\sum_{l=1}^{\infty} B_l\left[\int_0^{\pi} F_k(\theta)\sum_{m=0}^{l} b_{lm}\cos m\theta d\theta\right]^2\right.\\ &+\sum_{l=1}^{\infty}\frac{\pi}{a}\int_0^{\pi}\int_0^{\pi} F_k(\theta)F_k(\theta')\\ &\cdot\left[-\frac{\ln\alpha_2}{2}+\frac{\cos l\theta\cos l\theta'}{l}\right]d\theta d\theta'\\ &\left.+\frac{\gamma_k}{k^2}\left[\int_0^{\pi} F_k(\theta)\sum_{m=0}^{k} b_{km}\cos m\theta d\theta\right]^2\right\}\end{aligned}}{\frac{1}{k^2}\left[\int_0^{\pi} F_k(\theta)\sum_{m=0}^{k} b_{km}\cos m\theta d\theta\right]^2}. \tag{4}
$$

In (4), $B_l=(\gamma_l/l^2)-(\pi/la)$ and $F_k(\theta)\, d\theta/dx=dE_k(x)/dx$. Also, α_1 and α_2 are given by

$$\alpha_1 = \cos\left[\frac{\pi}{2a}(2x_1+d)\right]\cos\frac{\pi d}{2a} \tag{5a}$$

$$\alpha_2 = \sin\left[\frac{\pi}{2a}(2x_1+d)\right]\sin\frac{\pi d}{2a} \tag{5b}$$

and the b coefficients are defined by

$$\cos\frac{l\pi x}{a} = \sum_{m=0}^{l} b_{lm}\cos m\theta \tag{6a}$$

$$\cos\frac{\pi x}{a} = \alpha_1 + \alpha_2\cos\theta. \tag{6b}$$

A method of determining the b coefficients is given by Collin [8] along with a few examples for the symmetrical case ($\alpha_1=0$). A more complete list of values is given in Table I.

In general, for the TE_{k0} mode incident, we would have

$$F_k(\theta) = \sum_{c=0}^{\infty} K_c\cos c\theta. \tag{7}$$

This could be put into (4) and the stationary property of that equation used to determine Y_k. For practical purposes only a few terms of (7) could be retained in order to obtain numerical results. Since Y_k in (4) is relatively insensitive to the trial function, we will take just one term, the $c=k$ term.

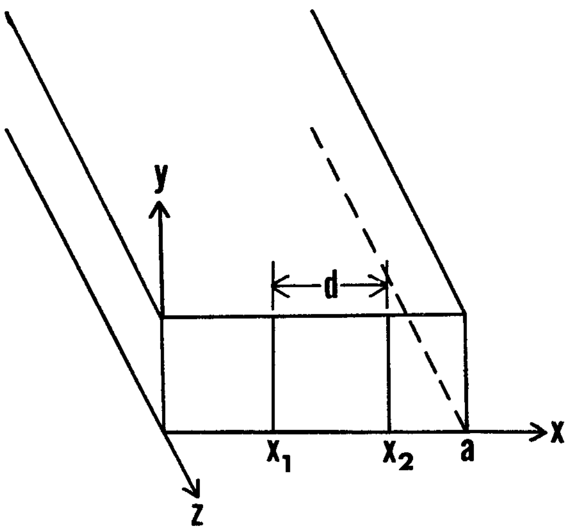

Fig. 1. Perfectly conducting iris in rectangular waveguide.

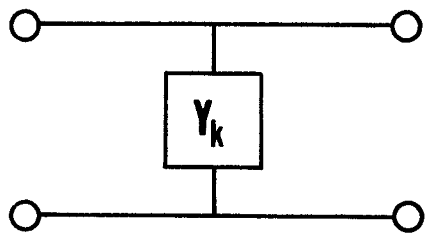

Fig. 2. Equivalent circuit of the iris discontinuity.

TABLE I
VALUES OF b_{nm}

n	b_{nn}	$b_{n(n-1)}$	$b_{n(n-2)}$	$b_{n(n-3)}$	$b_{n(n-4)}$
1	α_2	α_1			
2	α_2^2	$4\alpha_1\alpha_2$	$2\alpha_1^2+\alpha_2^2-1$		
3	α_2^3	$6\alpha_1\alpha_2^2$	$3\alpha_2(4\alpha_1^2+\alpha_2^2-1)$	$\alpha_1(4\alpha_1^2+6\alpha_2^2-3)$	
4	α_2^4	$8\alpha_1\alpha_2^3$	$4\alpha_2^2(6\alpha_1^2+\alpha_2^2-1)$	$\alpha_1\alpha_2(32\alpha_1^2+24\alpha_2^2-16)$	$1-8\alpha_1^2-4\alpha_2^2+8\alpha_1^4+3\alpha_2^4+24\alpha_1^2\alpha_2^2$
5	α_2^5	$10\alpha_1\alpha_2^4$	$5\alpha_2^3(8\alpha_1^2+\alpha_2^2-1)$	$\alpha_1\alpha_2^2(80\alpha_1^2+40\alpha_2^2-30)$	$\alpha_2(5-60\alpha_1^2-15\alpha_2^2+80\alpha_1^4+10\alpha_2^4+120\alpha_1^2\alpha_2^2)$
6	α_2^6	$12\alpha_1\alpha_2^5$	$6\alpha_2^4(10\alpha_1^2+\alpha_2^2-1)$	$\alpha_1\alpha_2^3(160\alpha_1^2+60\alpha_2^2-48)$	$\alpha_2^2(9-144\alpha_1^2-24\alpha_2^2+240\alpha_1^4+15\alpha_2^4+240\alpha_1^2\alpha_2^2)$
7	α_2^7	$14\alpha_1\alpha_2^6$	$7\alpha_2^5(12\alpha_1^2+\alpha_2^2-1)$	$\alpha_1\alpha_2^4(280\alpha_1^2+84\alpha_2^2-70)$	$\alpha_2^3(14-280\alpha_1^2-35\alpha_2^2+560\alpha_1^4+21\alpha_2^4+320\alpha_1^2\alpha_2^2)$

n	$b_{n(n-5)}$	$b_{n(n-6)}$
5	$\alpha_1(5-20\alpha_1^2-40\alpha_2^2+16\alpha_1^4+30\alpha_2^4+80\alpha_1^2\alpha_2^2)$	
6	$\alpha_1\alpha_2(36-192\alpha_1^2-124\alpha_2^2+192\alpha_1^4+120\alpha_2^4+480\alpha_1^2\alpha_2^2)$	$32\alpha_1^6-48\alpha_1^4+18\alpha_1^2-1+240\alpha_1^4\alpha_2^2-144\alpha_1^2\alpha_2^2+9\alpha_2^2+18\alpha_1^2\alpha_2^4-18\alpha_2^4+10\alpha_2^6$
7	$\alpha_1\alpha_2^2(168-1120\alpha_1^2-280\alpha_2^2+672\alpha_1^4+210\alpha_2^4+1120\alpha_1^2\alpha_2^2)$	$\alpha_2(448\alpha_1^6-560\alpha_1^4+168\alpha_1^2-7+1680\alpha_1^4\alpha_2^2-840\alpha_1^2\alpha_2^2+42\alpha_2^2+840\alpha_1^2\alpha_2^4-70\alpha_2^4+35\alpha_2^6)$

That is, assume $F_k(\theta)=K_k \cos k\theta$. For large apertures this assumed field distribution should be fairly accurate. For example, if $d=a$ (no iris present) this distribution is exact. As the aperture diminishes, this trial function probably becomes less accurate. The stationary characteristic of (4) will "absorb" part of the error. As will be shown in Section IV, the assumed trial function gives reasonable results when used in (4) even for the case of zero aperture ($d=0$).

Putting this trial function into (4) and setting $b_{kk}=\alpha_2^k$ gives

$$Y_k = \frac{2k^2}{\gamma_k \alpha_2^{2k}} \left[\sum_{l=1}^{\infty} B_l (b_{lk})^2 + \frac{\pi}{ka} - \frac{\gamma_k \alpha_2^{2k}}{k^2} \right]. \tag{8}$$

This equation is suitable for direct calculation of Y_k since the infinite summation converges quickly. Further analysis of (8) is performed in Section IV. This solution reduces to Collin's solution [9] if we take $k=1$ and assume a symmetrical pair of irises.

III. Coupling Coefficients

In this section the coupling coefficients are calculated explicitly. This is done by finding a variational expression for these terms directly. In this section we will be assuming that $k \neq l$.

A variational expression for R_{kl} is

$$\frac{\gamma_k}{R_{kl}} = \frac{\int_{AP}\int_{AP} E_l(x) E_k(x') G(x \mid x') dx dx'}{\int_{AP} E_l(x) \sin \frac{k\pi x}{a} dx \int_{AP} E_k(x') \sin \frac{l\pi x'}{a} dx'} \tag{9}$$

where $G(x \mid x')$ is defined by

$$G(x \mid x') = \sum_{n=1}^{\infty} \gamma_n \sin \frac{n\pi x}{a} \sin \frac{n\pi x'}{a} \tag{10}$$

and E_k and E_l are, respectively, the aperture fields when the TE_{k0} and TE_{l0} modes are incident. The value of R_{kl} in (9) depends on the functional form of E_l and E_k but not on their amplitudes.

It may be noted that since the right side of (9) is symmetrical in k and l, then

$$\frac{\gamma_k}{R_{kl}} = \frac{\gamma_l}{R_{lk}} \cdot \tag{11}$$

This is a form of the reciprocity relationship in the waveguide. Usually only the TE_{10} mode can propagate in the waveguide. If we take the case of $k>l$ and assume $k>2$, then $\gamma_k > \gamma_l$. Thus $R_{kl} > R_{lk}$. That is, the coupling from a high-order mode to a low-order mode is greater than the coupling from the low-order mode to the high-order mode.

Integrating by parts and making the same changes as in Section II results in (9) becoming

$$\frac{\gamma_k}{R_{kl}} = \frac{kl \int_0^{\pi}\int_0^{\pi} F_l(\theta) F_k(\theta') \sum_{n=1}^{\infty} \left[\frac{\pi}{a}\left(-\frac{\ln \alpha_2}{2} + \frac{\cos n\theta \cos n\theta'}{n} \right) + B_n \sum_{m=0}^{n} b_{nm} \cos m\theta \sum_{p=0}^{n} b_{np} \cos p\theta' \right] d\theta d\theta'}{\int_0^{\pi} F_l(\theta) \sum_{m=0}^{k} b_{km} \cos m\theta d\theta \int_0^{\pi} F_k(\theta) \sum_{m=0}^{l} b_{lm} \cos m\theta' d\theta'} . \tag{12}$$

In general, we could choose

$$F_l(\theta) = \sum_{d=0}^{N} L_d \cos d\theta, \quad F_k(\theta') = \sum_{c=0}^{M} K_c \cos c\theta' \tag{13}$$

where N and M are finite integers. The Ritz procedure could then be used to determine the unknown coefficients L_d and K_c and/or the value of R_{kl}. For the trial functions, assume

$$F_l(\theta) = L_l \cos l\theta + L_k \cos k\theta \tag{14a}$$

$$F_k(\theta') = K_k \cos k\theta' + K_l \cos l\theta'. \tag{14b}$$

One justification for using these trial functions is that reasonable results are obtained. It will be shown that these functions produce solutions for the coupling coefficients which agree with the known solutions for the cases of no iris ($d=0$) and a short circuit ($d=a$). The solutions also agree with the case of a symmetrical iris, where no coupling can exist between the kth and lth mode if $l-k$ is odd. Finally, the stationarity of (12) and the use of the Ritz procedure to eliminate the coefficients in (14) serve to smooth out errors due to differences between the actual fields and those postulated.

When (14) is used in (12) an algebraic expression for γ_k/R_{kl} is obtained in terms of the unknown coefficients K and L. The stationary nature of the result is taken advantage of by setting the partial derivatives of γ_k/R_{kl} with respect to the unknown coefficients equal to zero. The determinant of the resulting equations is then set equal to zero to obtain the coupling coefficient. The result is

$$\frac{\gamma_k}{R_{kl}} = -\frac{\begin{vmatrix} Q_{kk}' - \frac{\pi}{ka} & Q_{lk}' \\ Q_{kl}' & Q_{ll}' - \frac{\pi}{la} \end{vmatrix}}{\begin{vmatrix} Q_{kk} & Q_{kl} \\ Q_{kl}' & Q_{ll}' - \frac{\pi}{la} \end{vmatrix}} \qquad l > k \tag{15}$$

where

$$Q_{dc} = \frac{b_{lc}b_{kd}}{kl} \qquad Q_{dc}' = -\sum_{n=1}^{\infty} B_n b_{nd} b_{nc}. \tag{16}$$

The value of R_{kl} for $k>l$ is found by using (11) and (15). We can use (15) for direct calculation or further simplify it as in the following section.

IV. Quasi-Static Solutions

Simplifications in (8) and (15) can be obtained if we make the quasi-static approximation that[1] $\gamma_n = n\pi/a$ for $n>1$. Then $B_n=0$ for $n>1$ and (8) becomes

$$Y_k = \frac{2k^2}{\gamma_k \alpha_2^{2k}}\left[\left(\gamma_1 - \frac{\pi}{a}\right) b_{1k}^2 + \frac{\pi}{ka} - \frac{\gamma_k \alpha_2^{2k}}{k^2}\right]. \tag{17}$$

For $k=1$ this becomes

$$Y_1 = \frac{2\pi}{\gamma_1 a}\left\{\csc^2\left[\frac{\pi}{2a}(2x_1+d)\right]\csc^2\frac{\pi d}{2a} - 1\right\}. \tag{18}$$

This solution checks with that obtained by Ghose [10]. If the irises are symmetrical, then $2x_1+d=a$ and

$$Y_1 = \frac{2\pi}{\gamma_1 a}\left(\csc^2\frac{\pi d}{2a} - 1\right) = \frac{2\pi}{\gamma_1 a}\cot^2\frac{\pi d}{2a}\cdot \tag{19}$$

If $k>1$, then $b_{1k}=0$ and (17) becomes

$$Y_k = 2\left\{\csc^{2k}\left[\frac{\pi}{2a}(2x_1+d)\right]\csc^{2k}\frac{\pi d}{2a} - 1\right\} \qquad k>1. \tag{20}$$

If the irises are symmetrical, (20) becomes

$$R_{kk} = \sin^{2k}\frac{\pi d}{2a} - 1 \qquad k>1. \tag{21}$$

For the special case of no iris ($d=a$), we find that $R_{kk}=0$ as required. For the case where $d=0$ and a short circuit appears across the waveguide, then (21) correctly gives $R_{kk}=-1$. These results further justify the use of the trial function assumed.

Next, the quasi-static approximations will be used to simplify the expressions for the coupling coefficients. These approximations applied to (16) give $Q_{dc}'=0$ for d or c greater than unity. For d and c equal to unity, we have $Q_{11}' = -B_1(b_{11})^2$. Thus, for $k=1$, (15) gives

$$R_{1l} = \frac{b_{l1}}{l}\left[\frac{\gamma_1\alpha_2}{\frac{\pi}{a}(1-\alpha_2^2) + \gamma_1\alpha_2^2}\right] \qquad l>1. \tag{22}$$

If the iris is symmetrical, then $\alpha_1=0$ and $\alpha_2=\sin \pi d/2a$. Then, as seen from Table I, $b_{l1}=0$ for l even. Thus, $R_{1l}=0$ for l even, i.e., modes with odd symmetry about $x=a/2$ are not excited by the TE_{10} incident wave. This is the expected result due to the symmetry of the incident TE_{10} mode and the symmetry of the discontinuity. A few expressions for R_{1l} for l odd are given in Table II.

Using the equations in Table II, we can calculate the value of iris separation d required to suppress specific modes. $R_{13}=0$ only if $d=0$ or $d=a$. That is, if the irises form a complete short circuit or are omitted entirely, $R_{13}=0$. These two cases are evident from an exact analysis. $R_{15}=0$, if $d=0$, $d=a$, or $d=a/2$. The value $d=a/2$ is interesting as in this case the edges of both irises lie exactly halfway between a maximum and a zero of the electric field of the TE_{50} mode. $R_{17}=0$ if $d=0$, $d=a$, $d=0.354a$, or $d=0.645a$. For $d=0.354a$ and $d=0.645a$ the edges of the irises again lie exactly between a maximum and a zero of the TE_{70} electric field.

It is interesting to calculate the total electric field at the $z=0$ plane as predicted by (1) with the value of R_{11} obtained from (19) and the values of R_{13}, R_{15}, and R_{17} obtained from Table II. The results are plotted in Fig. 3 for the case when $d=0.75a$. The close agreement with the expected result of zero field over the iris gives added confidence to the method and choice of trial functions. The maximum deviation from the known result is at the iris edge ($x=0.125a$ and $x=0.875a$). The error at this point can be described by the ratio of the field at the edge to the field at the center of the waveguide ($x=a/2$). The error in this case is 11.5 percent.

TABLE II

Values of R_{1l} for TE_{10} Mode Incidence on a Symmetrical Pair of Irises. Quasi-Static Solution. $\alpha_2 = \sin \pi d/2a$

R_{13}	$=$	$\dfrac{a\gamma_1\alpha_2^2(\alpha_2^2-1)}{\pi(1-\alpha_2^2)+a\gamma_1\alpha_2^2}$
R_{15}	$=$	$(2\alpha_2^2-1)R_{13}$
R_{17}	$=$	$(5\alpha_2^4-5\alpha_2^2+1)R_{13}$

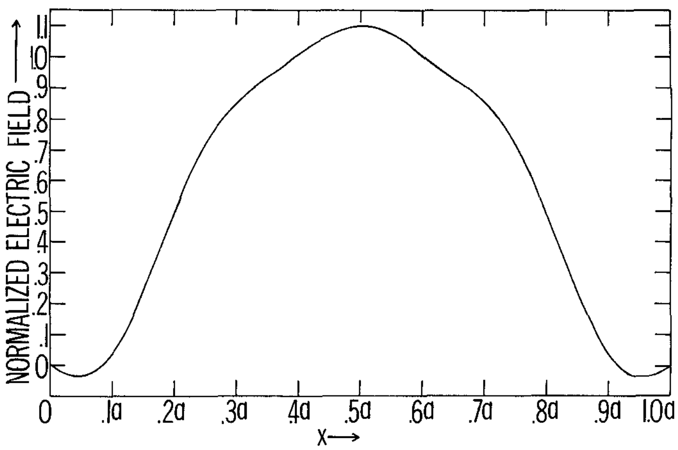

Fig. 3. Electric field at the iris discontinuity for $d=0.75a$ using the quasi-static solutions for the reflection and coupling coefficients. The TE_{10} mode is incident.

[1] This approximation is suitable when only the TE_{10} mode can propagate in the waveguide. Then γ_n is real for $n>1$. (If higher-order modes are allowed to propagate, we could let $\gamma_n = n\pi/a$ for $n>l$, where the TE_{l0} mode is the highest-order propagating mode.)

A very small change in the results of Fig. 3 are obtained if we include modes higher than the TE_{70} mode in (1). If we use (8) and (15) instead of the quasi-static approximations the error is reduced a small amount (to 10.3 percent). More accurate results can be obtained by including more terms in the trial functions.

Finally, the quasi-static approximations can be applied to (15) with $k>1$. The result is

$$R_{kl} = \frac{kb_{lk}b_{kk}}{l} \qquad k > 1, l > k. \tag{23}$$

An analysis of (23) using Table I shows that $R_{kl}=0$ if $d=a$ or $d=0$. R_{kl} is also zero if the iris is symmetrical and $l-k$ is odd. This correspondence with known results adds confidence to the use of the trial functions (14).

V. Summary

The variational technique permits us to find the reflection coefficient and the coupling coefficients independently. In order to obtain equations for the reflection and coupling coefficients easily, relatively simple trial functions can be chosen for use in the variational expressions. The results obtained in the text reduce to known solutions for the degenerate cases of no iris ($d=0$) and a short circuit ($d=a$). If more accurate results are required, more terms in the trial functions may be included. From the reflection and coupling coefficients we can construct the entire field at any point in the waveguide. Particularly interesting is the field at the plane of the aperture. This field distribution can be calculated with errors on the order of 12 percent.

Certain modes can be suppressed from the reflected waves. This could be useful in some application where a particular mode is harmful. Alternatively, it might be desirable to excite some particular mode. The analysis could be used to find how to do this efficiently.

From (11) we see that the coupling from a high-order mode to a low-order mode is greater than the coupling from a low-order mode to the high-order one.

The derivations in the paper apply to cases where any number of TE_{k0} modes may propagate in the waveguide. Numerical results were obtained for the case where only the TE_{10} mode was propagating but suggestions were made for including multimode propagation. Results of this type would be applicable to propagation in oversized rectangular waveguide such as that used at millimeter wavelengths.

References

[1] L. Lewin, *Advanced Theory of Waveguides*. London: Iliffe and Sons, Ltd., 1951.
[2] R. N. Ghose, *Microwave Circuit Theory and Analysis*. New York: McGraw-Hill, 1963.
[3] N. Marcuvitz, *Waveguide Handbook*. New York: McGraw-Hill, 1951.
[4] R. E. Collin, *Field Theory of Guided Waves*. New York: McGraw-Hill, 1960.
[5] J. Schwinger, "The mathematical analyses of waveguide discontinuities," unpublished notes.
[6] R. E. Collin [4], pp. 348–352.
[7] *Ibid.*, p. 349.
[8] *Ibid.*, pp. 343–344, 351.
[9] *Ibid.*, p. 351.
[10] R. N. Ghose [2], p. 324.